

ON THE NEWTON POLYGONS OF ABELIAN VARIETIES OF MUMFORD'S TYPE

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ABSTRACT. Let A be an abelian variety of Mumford's type. This paper determines all possible Newton polygons of A in char p . This work generalizes a result of R. Noot in [7]-[8].

1. INTRODUCTION

In [5] D. Mumford constructs families of abelian varieties over smooth projective arithmetic quotients of the upper half plane, whose general fiber has only \mathbb{Z} as its endomorphism ring. This is the first example of Shimura families of Hodge type. Roughly speaking, a Shimura family of Hodge type is characterized by the Hodge classes on the tensor products of the weight one \mathbb{Q} -polarized Hodge structure of an abelian variety and its dual. When they are generated by $(1, 1)$ -classes, it is called a Shimura curve of PEL type: By the Lefschetz $(1, 1)$ theorem, they are classes of algebraic cycles. Since the Hodge conjecture or its alike has not been settled in general, it is often the case that a result can be established relatively easily for the PEL type than for the Hodge type.

In a series of papers [6],[7],[8] R. Noot studies the Galois representations associated to an abelian variety over a number field which appears as a closed fiber of a Mumford's family, and he obtains various results about such an abelian variety, notably potentially good reduction, classification and existence of isogeny types and Newton polygons. Some of the arguments in loc. cit. work also in a general setting.

From a completely different direction, E. Viehweg and the second named author characterize those families of abelian varieties which reach the Arakelov bound (cf. [11]), and it turns out that under a natural assumption one can classify them up to isogeny and finite étale base change. The results are closely related with Mumford's families and we restate them here as follows:

Theorem 1.1 (Viehweg-Zuo, Theorem 0.2, Theorem 0.5, [11]). *Let $f : X \rightarrow Y$ be a semi-stable family of g -dimensional abelian varieties, smooth over $U := Y - S$. Assume that there is no unitary local subsystem in $R^1 f_* \mathbb{C}_{f^{-1}U}$. Then if f reaches the Arakelov bound, namely the following Arakelov inequality*

$$2 \cdot \deg(f_* \Omega_{X|Y}(\log f^{-1}(S))) \leq g \cdot \deg(\Omega_Y^1(\log S))$$

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becomes equality, then after a possible finite étale base change, f is isogenous to

Case 1: the g -fold self product of a universal semi-stable family of elliptic curves over a modular curve if $S \neq \emptyset$,

Case 2: the self product of a Shimura family of Mumford type if $S = \emptyset$.

The latter case generalizes the example of Mumford in a natural way. The construction in loc. cit. will be briefly recalled (see §1 for more details): Let F be a totally real field of degree $d \geq 1$ and D a quaternion division algebra over F , which is split only at one real place of F . The corestriction $\text{Cor}_{F|\mathbb{Q}} D$ is isomorphic to $M_{2d}(\mathbb{Q}(\sqrt{b}))$ for a rational number b . Depending on whether b is a square or not, one is able to construct Shimura families of Hodge type with the fiber dimension $2^{d+\epsilon(D)-1}$. The cases $d = 1, 2$ give rise to Shimura families of PEL type, and the case $d = 3$ is just the example of Mumford.

By the theory of canonical models of Shimura varieties, these families are defined naturally over certain number fields. By abuse of notation, we call a \mathbb{Q} -closed fiber of such a Shimura family an abelian variety of *Mumford's type*. The purpose of the paper is to study the p -adic Galois representation of an abelian variety of Mumford's type. That is, we intend to study the p -adic Hodge structure of such an abelian variety using the p -adic Hodge theory of Fontaine and his school. Certainly there will be some overlaps of our approach with what Noot has done in his study on Mumford's example (see particularly §3 [7], §2 [8]). However we shall emphasize that the relation between two dimensional potentially crystalline \mathbb{Q}_p -representations and p -adic Hodge structures of abelian varieties of Mumford's type is essentially new. A first study of this relation gives us the classification of Newton polygons of good reductions of abelian varieties of Mumford's type. To make the result complete, we also include the existence of possible Newton polygons, which is based on Noot §3-5 [8].

Theorem 1.2. *Let F be a totally real field of degree $d \geq 1$ and D a quaternion division algebra over F , which is split only at one real place of F . Let A be an abelian variety of Mumford's type associated to D , which is defined over a number field $K \supset F$. Let p be a rational prime number satisfying Assumption 2.2 and \mathfrak{p}_K a prime of K over p . Assume that A has the good reduction A_k at \mathfrak{p}_K . Put $\mathfrak{p} = \mathfrak{p}_K \cap \mathcal{O}_F$ and $r = [F_{\mathfrak{p}} : \mathbb{Q}_p]$. Then the Newton polygon of A_k is either $\{2^{d+\epsilon(D)} \times \frac{1}{2}\}$ (i.e. supersingular) or*

$$\{2^{d-r+\epsilon(D)} \times 0, \dots, 2^{d-r+\epsilon(D)} \cdot \binom{r}{i} \times \frac{i}{r}, \dots, 2^{d-r+\epsilon(D)} \times 1\}.$$

Here $\epsilon(D)$ is equal to 0 or 1 depending only on D (see §1 for the definition). Furthermore each possible Newton polygon does occur for an abelian variety of Mumford's type associated to D .

The above result confirms the conjecture on the Newton stratification of a Shimura curve of Hodge type made in [9] partially.

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2. ABELIAN VARIETIES OF MUMFORD'S TYPE AND TENSOR DECOMPOSITION OF GALOIS REPRESENTATIONS

Let us start with the following

Lemma 2.1 (Lemma 5.7 (a) [11]). *Let F be a totally real field of degree d and D be a quaternion division algebra over F , which is split at a unique real place of F . Let $\text{Cor}_{F|\mathbb{Q}}(D)$ be the corestriction of D to \mathbb{Q} . Then*

- (i) $\text{Cor}_{F|\mathbb{Q}}(D) \cong M_{2d}(\mathbb{Q})$ and d is odd, or
- (ii) $\text{Cor}_{F|\mathbb{Q}}(D) \not\cong M_{2d}(\mathbb{Q})$. Then

$$\text{Cor}_{F|\mathbb{Q}}(D) \cong M_{2d}(\mathbb{Q}(\sqrt{b})),$$

where $\mathbb{Q}(\sqrt{b})$ is a(n) real (resp. imaginary) quadratic field extension of \mathbb{Q} if d is odd (resp. even).

For a given D , an isomorphism in the above lemma will be fixed once for all. Clearly we can write either case $\text{Cor}_{F|\mathbb{Q}}(D) \cong M_{2d}(\mathbb{Q}(\sqrt{b}))$ for a rational number $b \in \mathbb{Q}$. If we require b to be square free, then b is uniquely determined by D . We define a function $\epsilon(D)$ which takes the value 0 when $b = 1$ and 1 otherwise. For case (ii) we shall also further choose an embedding $M_{2d}(\mathbb{Q}(\sqrt{b})) \hookrightarrow M_{2d+1}(\mathbb{Q})$. So we shall fix an embedding $\text{Cor}_{F|\mathbb{Q}}(D) \hookrightarrow M_{2d+\epsilon(D)}(\mathbb{Q})$ in either case.

Let $\bar{}$ be the standard involution of D . One defines a \mathbb{Q} -simple group

$$\tilde{G}' := \{x \in D \mid x\bar{x} = 1\},$$

and $\tilde{G} := \mathbb{G}_{m,\mathbb{Q}} \times \tilde{G}'$. Recall that there is a natural morphism of \mathbb{Q} -groups (cf. §4 [5])

$$\text{Nm} : D^* \rightarrow \text{Cor}_{F|\mathbb{Q}}(D)^* \hookrightarrow \text{GL}_{2d+\epsilon(D)}(\mathbb{Q}).$$

One defines a reductive \mathbb{Q} -group G to be the image of the morphism $\tilde{G} \rightarrow \text{GL}_{2d+\epsilon(D)}(\mathbb{Q})$, that is the product of the natural morphism $\mathbb{G}_{m,\mathbb{Q}} \rightarrow \text{GL}_{2d+\epsilon(D)}(\mathbb{Q})$ and $\text{Nm}|_{\tilde{G}'}$. The image of \tilde{G}' in G is denoted by G' . The resulting natural morphism

$$N : \tilde{G} = \mathbb{G}_{m,\mathbb{Q}} \times \tilde{G}' \rightarrow G$$

is a central isogeny. The set of real places of F is denoted by $\{\tau_i\}_{1 \leq i \leq d}$, and we assume that D is split over $\tau := \tau_1$. Then one has an isomorphism of real groups:

$$\tilde{G}'(\mathbb{R}) \simeq \text{SL}_2(\mathbb{R}) \times \text{SU}(2)^{\times d-1}.$$

One defines

$$u_0 : S^1 \rightarrow \tilde{G}'(\mathbb{R}), \quad e^{i\theta} \mapsto \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \times id^{\times d-1},$$

and $\tilde{h}_0 = id \times u_0 : \mathbb{R}^* \times S^1 \rightarrow \tilde{G}(\mathbb{R})$ descends to a morphism of real groups:

$$h_0 : \mathbb{S} \rightarrow G_{\mathbb{R}}.$$

Let X be the $G(\mathbb{R})$ -conjugacy class of h_0 , and one verifies that (G, X) gives the Shimura datum of a Shimura curve. Let $W_{\mathbb{Q}} = \mathbb{Q}(\sqrt{b})^{2d}$ with the natural action of G' . It is easy to verify that there exists a unique $\mathbb{Q}(\sqrt{b})$ -valued symplectic form ω on $W_{\mathbb{Q}}$ up to scalar invariant under G' -action. One defines a \mathbb{Q} -valued symplectic form $\psi = Tr_{\mathbb{Q}(\sqrt{b})|\mathbb{Q}}\omega$ on $W_{\mathbb{Q}}$ and $(\mathrm{GSp}(W_{\mathbb{Q}}, \psi), X(\psi))$ be the Siegel modular variety. Then there is a natural embedding $G \hookrightarrow \mathrm{GSp}(W_{\mathbb{Q}}, \psi)$ which carries X into $X(\psi)$: This realizes (G, X) as a Shimura curve of Hodge type. Let $C \subset G(\mathbb{A}_f)$ be a compact open subgroup and one defines the Shimura curve as the double coset

$$Sh_C(G, X) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / C,$$

where

$$q(x, a)b = (qx, qab), \quad q \in G(\mathbb{Q}), x \in X, a \in G(\mathbb{A}_f), b \in C.$$

The theory of canonical model says that $Sh_C(G, X)$ is naturally defined over its reflex field $\tau(F) \subset \mathbb{C}$. By choosing C small enough, there exists a universal family of abelian varieties $\mathcal{X}_C \rightarrow Sh_C(G, X)$ which is also defined over $\tau(F)$. A Shimura family of Mumford type in Theorem 1.1 is a geometrically connected component of such a universal family (different components are isomorphic to each other over $\bar{\mathbb{Q}}$). We call a $\bar{\mathbb{Q}}$ -closed fiber of such a Shimura family an abelian variety of Mumford's type.

We take the following assumption on the prime number p :

Assumption 2.2. Let p be a rational prime which does not divide the discriminants of F and D .

we have then

$$p\mathcal{O}_F = \prod_{i=1}^n \mathfrak{p}_i.$$

Let $\bar{\mathbb{Q}}$ be the algebraic closure of \mathbb{Q} in \mathbb{C} . By choosing an embedding $\bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_p$, one gets an identification

$$\mathrm{Hom}_{\mathbb{Q}}(F, \bar{\mathbb{Q}}) = \mathrm{Hom}_{\mathbb{Q}}(F, \bar{\mathbb{Q}}_p) = \prod_{i=1}^n \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \bar{\mathbb{Q}}_p).$$

In the rest of the paper such an embedding will be fixed. By a possible rearrangement of indices, we can assume that $\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_1}, \bar{\mathbb{Q}}_p)$ under the above identification. By the above assumption one has an isomorphism

$$D \otimes_F F_{\mathfrak{p}_1} \simeq M_2(F_{\mathfrak{p}_1}).$$

Such an isomorphism will be also fixed. Put $r = [F_{p_1} : \mathbb{Q}_p]$.

Let A be an abelian variety of Mumford's type defined over a number field K . Let $\rho : \text{Gal}_K \rightarrow \text{GL}(H_{\text{et}}^1(\bar{A}, \mathbb{Q}_p))$ be the associated Galois representation. For simplicity we write $H_{\mathbb{Q}_p}$ for $H_{\text{et}}^1(\bar{A}, \mathbb{Q}_p)$. The aim of this section is to show the following tensor decomposition:

Proposition 2.3. *Restricted to an open subgroup $\text{Gal}_{K'} \subset \text{Gal}_K$, $(\rho, H_{\mathbb{Q}_p})$ has a natural tensor decomposition:*

$$H_{\mathbb{Q}_p} \simeq (V_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p})^{\oplus 2^{\epsilon(D)}},$$

together with a further tensor decomposition of $V_{\mathbb{Q}_p}$:

$$V_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \simeq V_1 \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^{r-1}},$$

where for $0 \leq i \leq r-1$, V_{1,σ^i} is the σ^i -conjugate of the two dimensional \mathbb{Q}_{p^r} -representation V_1 .

Proof. Let $H_{\mathbb{Q}} = H_B^1(A(\mathbb{C}), \mathbb{Q})$ be the singular cohomology, and $G_A \subset \text{GL}(H_{\mathbb{Q}})$ be the Mumford-Tate group of A (strictly speaking it is the projection of the Mumford-Tate group of A to the first factor of $\text{GL}(H_{\mathbb{Q}}) \times \mathbb{G}_{m,\mathbb{Q}}$ cf. §3, [2]). By construction of the Shimura curve $Sh_C(G, X)$, under a natural identification $H_{\mathbb{Q}} = W_{\mathbb{Q}}$ the Mumford-Tate group G_A is a subgroup of $\xi : G \hookrightarrow \text{GL}(W_{\mathbb{Q}})$. By Proposition 2.9 (b) in loc. cit., there is a finite field extension $K \subset K' \subset \bar{\mathbb{Q}}$ such that the representation

$$\rho|_{\text{Gal}_{K'}} : \text{Gal}_{K'} \rightarrow \text{GL}(H_{\mathbb{Q}_p}) = \text{GL}(W_{\mathbb{Q}_p})$$

factors through

$$\xi_{\mathbb{Q}_p} : G(\mathbb{Q}_p) \hookrightarrow \text{GL}(W_{\mathbb{Q}_p}).$$

In the following we examine the morphism $\xi_{\mathbb{Q}_p}$ closely. One has a short exact sequence of \mathbb{Q} -groups:

$$1 \rightarrow G' \rightarrow G \rightarrow \mathbb{G}_m \rightarrow 1.$$

In order to show the tensor decomposition it suffices to consider the restriction of ξ to the subgroup G' and its base change to \mathbb{Q}_p . Recall that $\text{Cor}_{F|\mathbb{Q}}(D)$ is defined as the $\text{Gal}_{\mathbb{Q}}$ -invariants of the $\bar{\mathbb{Q}}$ -algebra $\otimes_{i=1}^d D_{\tau_i}$ where $D_{\tau_i} = D \otimes_F (\bar{\mathbb{Q}}, \tau_i)$. Since D_{τ_i} is split, it acts on $H_i = D_{\tau_i} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by the left multiplication. It is isomorphic to the standard representation of $M_2(\bar{\mathbb{Q}})$ on $\bar{\mathbb{Q}}^2$. Then as $\text{Cor}_{F|\mathbb{Q}}(D) \otimes \bar{\mathbb{Q}}$ -modules one has a natural tensor decomposition

$$W_{\mathbb{Q}} \otimes \bar{\mathbb{Q}} = (\otimes_{i=1}^d H_i)^{\oplus 2^{\epsilon(D)}}.$$

The decomposition group $D_p \subset \text{Gal}_{\mathbb{Q}}$, defined by the chosen embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, is isomorphic to the local Galois group $\text{Gal}_{\mathbb{Q}_p}$. It acts on the set $\{\tau_i\}_{1 \leq i \leq d}$ and by Assumption 2.2 it decomposes into n orbits according to the prime decomposition. WLOG we can write $\{\tau = \tau_1, \dots, \tau_r\}$ for the orbit containing τ . Put then $V_{\bar{\mathbb{Q}}_p} = \otimes_{i=1}^r (H_i \otimes_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}_p)$ and $U_{\bar{\mathbb{Q}}_p} = \otimes_{i=r+1}^d (H_i \otimes_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}_p)$. Thus one has a natural tensor decomposition

$$W_{\mathbb{Q}} \otimes \bar{\mathbb{Q}}_p = (V_{\bar{\mathbb{Q}}_p} \otimes U_{\bar{\mathbb{Q}}_p})^{\oplus 2^{\epsilon(D)}}.$$

Taking the $\text{Gal}_{\mathbb{Q}_p}$ -invariants, one obtains a tensor decomposition as $\text{Cor}_{F|\mathbb{Q}}(D) \otimes \mathbb{Q}_p$ -modules

$$W_{\mathbb{Q}_p} = (V_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p})^{\oplus 2^{\epsilon(D)}}.$$

Since the action of $G''(\mathbb{Q}_p)$ on $W_{\mathbb{Q}_p}$ via $\xi_{\mathbb{Q}_p}$ factors through that of $\text{Cor}_{F|\mathbb{Q}}(D) \otimes \mathbb{Q}_p$, the above decomposition is also a tensor decomposition for the $\text{Gal}_{K'}$ action. To proceed further, we consider the base change of

$$\text{Nm} : D^* \rightarrow \text{Cor}_{F|\mathbb{Q}}(D)^*, \quad d \mapsto (d \otimes 1) \otimes \cdots \otimes (d \otimes 1).$$

to \mathbb{Q}_p . Let F_i be the completion of F at the prime \mathfrak{p}_i . Then one has a natural isomorphism $\text{Nm} \otimes \mathbb{Q}_p = \prod_{i=1}^n \text{Nm}_i$ with

$$\text{Nm}_i : (D \otimes_F F_i)^* \rightarrow \text{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i)^*.$$

By Assumption 2.2, $F_1 \simeq \mathbb{Q}_{p^r}$ and $D \otimes_F F_1 \simeq M_2(\mathbb{Q}_{p^r})$. It is clear that the action of $\text{Gal}_{\mathbb{Q}_p}$ on

$$\text{Cor}_{F_1|\mathbb{Q}_p}(D \otimes_F F_1) \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p \simeq [M_2(\mathbb{Q}_{p^r}) \otimes_{\mathbb{Q}_{p^r}} (\bar{\mathbb{Q}}_p, \tau_1)] \otimes \cdots \otimes [M_2(\mathbb{Q}_{p^r}) \otimes_{\mathbb{Q}_{p^r}} (\bar{\mathbb{Q}}_p, \tau_r)]$$

factors through

$$\text{Gal}_{\mathbb{Q}_p} \twoheadrightarrow \text{Gal}_{\mathbb{Q}_{p^r}|\mathbb{Q}_p} = \langle \sigma \rangle.$$

Thus it descends to the tensor decomposition

$$\text{Cor}_{F_1|\mathbb{Q}_p}(D \otimes_F F_1) \otimes_{\mathbb{Q}_p} F_1 = \otimes_{i=0}^{r-1} [M_2(F_1) \otimes_{F_1} (F_1, \sigma^i)] = \otimes_{i=0}^{r-1} M_2(F_1^{\sigma^i}).$$

This gives us the further tensor decomposition of $V_{\mathbb{Q}_p}$. □

3. TWO DIMENSIONAL POTENTIALLY CRYSTALLINE \mathbb{Q}_{p^r} -REPRESENTATIONS

A large portion of this section is expository and it is based on §7.3 [3]. First we recall several standard notions in the theory of p -adic representations (see for example [3]). Let E be a finite field extension of \mathbb{Q}_p and V a p -adic representation of Gal_E . Let B be one of the period rings $B_{HT}, B_{dR}, B_{crys}, B_{st}$ introduced by Fontaine.

Definition 3.1. A representation V is called B -admissible if the tensor representation $V \otimes B$ of Gal_E is trivial, namely there exists a B -basis of $V \otimes B$ on which Gal_E acts trivially.

Let $D_B(V) = (V \otimes B)^{\text{Gal}_E}$ which is naturally a B^{Gal_E} -vector space. A nice property is that one has always $\dim_{B^{\text{Gal}_E}} D_B(V) \leq \dim_{\mathbb{Q}_p} V$, and the equality holds iff V is B -admissible (See Theorem 2.13 in [3]). In each of three cases, one calls V respectively *Hodge-Tate*, *de Rham*, *crystalline* and *semi-stable* (or *log crystalline*) when V is B_{HT} (resp. B_{dR}, B_{crys}, B_{st})-admissible. Furthermore, if the B -admissibility of V holds only for an open subgroup of Gal_E , then one calls V potentially B -admissible.

The following simple lemma precedes a further discussion.

Lemma 3.2. *Let V be a B -admissible representation of Gal_E and $E' \subset \bar{\mathbb{Q}}_p$ a finite field extension of E . Regarded as $\text{Gal}_{E'}$ -representation via restriction, V is then still B -admissible. Moreover for a crystalline representation V , one has the equality of filtered ϕ -modules:*

$$(D_{\text{crys},E}(V), \phi, \text{Fil}) = (D_{\text{crys},E'}(V) \otimes_{E'_0} E_0, \phi \otimes \sigma, \text{Fil}).$$

Proof. By $D_{\text{crys},E} = D_{\text{crys},E'}^{\text{Gal}(E'|E)}$ and the injectivity of the map

$$\alpha : D_{\text{crys},E'}^{\text{Gal}(E'|E)} \otimes_{E'_0} E_0 \rightarrow D_{\text{crys},E},$$

one finds that V is crystalline as $\text{Gal}_{E'}$ -representation and α is an isomorphism. \square

Let $r \in \mathbb{N}$. We recall the original definition of \mathbb{Q}_{p^r} -representation in §7.3 [3].

Definition 3.3. A \mathbb{Q}_{p^r} -representation of Gal_E is a finite dimensional \mathbb{Q}_{p^r} -vector space V equipped with a continuous action

$$\text{Gal}_E \times V \rightarrow V$$

satisfying

$$g(v_1 + v_2) = g(v_1) + g(v_2), \quad g(\lambda v) = g(\lambda)g(v)$$

where $g \in \text{Gal}_E$, and $\lambda \in \mathbb{Q}_{p^r}$, $v, v_1, v_2 \in V$

Note after choosing a \mathbb{Q}_{p^r} -basis of V , one gets a map

$$f : \text{Gal}_E \rightarrow \text{GL}_n(\mathbb{Q}_{p^r}).$$

However the map f is generally *not* a group homomorphism. Rather it gives an element in

$$Z_{\text{cts}}^1(\text{Gal}_E, \text{GL}_n(\mathbb{Q}_{p^r})) = \{f(g_1 g_2) = f(g_1)g_1(f(g_2)), \quad f \text{ continuous}\},$$

where Gal_E acts on $\text{GL}_n(\mathbb{Q}_{p^r})$ by acting on the entries. Its isomorphism class is an element in the non-abelian group cohomology $H_{\text{cts}}^1(\text{Gal}_E, \text{GL}_n(\mathbb{Q}_{p^r}))$. Note that the action of Gal_E on \mathbb{Q}_{p^r} factors through the quotient $\text{Gal}_{k_E} = \langle \phi_E^f \rangle$ where k_E is the residue field of E and $f_E = [k_E : \mathbb{F}_p]$ is the residue degree of E . This implies that the image of Gal_E on $\text{Gal}(\mathbb{Q}_{p^r}|\mathbb{Q}_p)$ is generated by $\sigma^{f_E \bmod r}$. Thus one finds that the action of Gal_E on \mathbb{Q}_{p^r} is trivial iff f_E is a multiple of r , or equivalently E contains \mathbb{Q}_{p^r} . This fact will simplify our consideration. The following lemma follows also easily from the discussion.

Lemma 3.4. *Let (V, ρ) be a \mathbb{Q}_p -representation of Gal_E . Let $r \in \mathbb{N}$ such that $\mathbb{Q}_{p^r} \subset E$. Then (V, ρ) is a \mathbb{Q}_{p^r} -representation iff there is an injection of \mathbb{Q}_p -algebras $\mathbb{Q}_{p^r} \hookrightarrow \text{End}_{\mathfrak{g}}(V)$, where \mathfrak{g} is the Lie algebra of the p -adic Lie group $\rho(\text{Gal}_E) \subset \text{GL}(V)$.*

Definition 3.5. A \mathbb{Q}_{p^r} -representation V is called B -admissible if it is B -admissible as \mathbb{Q}_p -representation.

For a \mathbb{Q}_{p^r} -representation V and each $0 \leq m \leq r-1$, one puts

$$V_{\sigma^m} := \mathbb{Q}_{p^r} \otimes_{\sigma^m, \mathbb{Q}_{p^r}} V,$$

together with tensor product of Gal_E -action. The notation \otimes_{σ^m} signifies the equalities of two tensors:

$$\lambda(\mu \otimes x) = \lambda\mu \otimes x, \quad \lambda \otimes \mu x = \lambda\mu^{\sigma^m} \otimes x.$$

Lemma 3.6. *Let $f : \text{Gal}_E \rightarrow \text{GL}_n(\mathbb{Q}_{p^r})$ be an element in $Z_{\text{cts}}^1(\text{Gal}_E, \text{GL}_n(\mathbb{Q}_{p^r}))$ corresponding to V . Then the map f^{σ^m} , which is defined by*

$$g \mapsto (f(g))^{\sigma^m},$$

is again an element in $Z_{\text{cts}}^1(\text{Gal}_E, \text{GL}_n(\mathbb{Q}_{p^r}))$, and in fact it corresponds to V_{σ^m} .

Proof. Let e_1, \dots, e_n be a basis of V . $\tilde{e}_i := 1 \otimes e_i$ basis of V_{σ^m} . $g(e) = f(g)e$. So

$$g(\tilde{e}) = 1 \otimes g(e) = 1 \otimes f(g)e = ((f(g))^{\sigma^m} 1) \otimes e = f^{\sigma^m}(g)\tilde{e}.$$

□

Lemma 3.7. *For each m , V_{σ^m} is isomorphic to V as \mathbb{Q}_p -representation. In particular, V is B -admissible iff V_{σ^m} is B -admissible.*

Proof. One verifies that the map

$$V \rightarrow V_{\sigma^m}, \quad v \mapsto 1 \otimes v$$

is \mathbb{Q}_p -linear isomorphism and Gal_E -equivariant. □

However they are not isomorphic as \mathbb{Q}_{p^r} -representations as we will see. For that one studies the invariants $\{D_{\text{crys}, r}^{(m)}(V)\}_{0 \leq m \leq r-1}$ introduced for a \mathbb{Q}_{p^r} -representation V .

Definition 3.8. Let V be a \mathbb{Q}_{p^r} -representation. For each $0 \leq m \leq r-1$, one defines

$$D_{B, r}^{(m)}(V) := (B \otimes_{\sigma^m, \mathbb{Q}_{p^r}} V)^{\text{Gal}_E}.$$

The following trivial lemma underlies the natural direct sum decomposition for a \mathbb{Q}_{p^r} -representation V :

Lemma 3.9. *One has a natural isomorphism of Gal_E -representations:*

$$\mathbb{Q}_{p^r} \otimes_{\mathbb{Q}_p} V \simeq \bigoplus_{m=1}^{r-1} \mathbb{Q}_{p^r} \otimes_{\sigma^m, \mathbb{Q}_{p^r}} V$$

Proof. One defines the map

$$a \otimes v \mapsto (av, \dots, a^{\sigma^m} v, \dots, a^{\sigma^{r-1}} v) := f(a \otimes v).$$

It is injective and \mathbb{Q}_{p^r} -linear and hence bijective. One also checks that $f(ga \otimes gv) = g(f(a \otimes v))$ which follows from the trivial fact that Gal_E acts on \mathbb{Q}_{p^r} via a power of σ as discussed above. □

From the lemma one has for example the natural decomposition:

$$B_{crys} \otimes_{\mathbb{Q}_p} V \simeq (B_{crys} \otimes_{\mathbb{Q}_{p^r}} \mathbb{Q}_{p^r}) \otimes_{\mathbb{Q}_p} V \simeq B_{crys} \otimes_{\mathbb{Q}_{p^r}} (\mathbb{Q}_{p^r} \otimes_{\mathbb{Q}_p} V) \simeq \bigoplus_{m=0}^{r-1} B_{crys} \otimes_{\sigma^m, \mathbb{Q}_{p^r}} V.$$

This implies that one has the decomposition of E_0 -vector spaces:

$$D_{crys}(V) = \bigoplus_{m=0}^{r-1} D_{crys,r}^{(m)}(V).$$

Certainly we shall discuss the relation of this decomposition to the σ -linear map ϕ and the filtration Fil on $D_{dR}(V) = D_{crys}(V) \otimes_{E_0} E$.

Lemma 3.10. *The map ϕ permutes the direct factors $D_{crys,r}^{(m)}(V)$ cyclically. Consequently, one has the decomposition of ϕ^r -modules:*

$$(D_{crys,r}(V), \phi^r) = \bigoplus_{m=0}^{r-1} (D_{crys,r}^{(m)}(V), \phi^r|_{D_{crys,r}^{(m)}(V)}).$$

Moreover, each ϕ^r -submodule $(D_{crys,r}^{(m)}(V), \phi^r)$ has the same Newton slopes.

Proof. Let $d = b \otimes_{\sigma^m} v \in D^{(m)}$, from the formula

$$\phi(d) = \phi(b) \otimes_{\sigma^{m+1} \bmod r} v,$$

we see that $\phi(d) \in D^{(m+1 \bmod r)}$. Since ϕ is bijective (so is ϕ^r which maps the summand $D^{(m)}$ to itself) and σ -linear (which can be viewed as linear map between two *different* vector spaces), the map

$$\phi : D_{crys,r}^{(m)}(V) \rightarrow D_{crys,r}^{(m+1 \bmod r)}(V)$$

is bijective and σ -linear isomorphism.

If $(f_E, r) = l$, then one can write $af_E = l + br$ with $a, b \in \mathbb{N}$. Thus ϕ^{af_E} is actually linear and induces an isomorphism $D_{crys,r}^{(m)}(V) \simeq D_{crys,r}^{(m+l \bmod r)}(V)$ of ϕ^r -modules. In particular, if f_E and r is coprime, then each direct factor $(D_{crys,r}^{(m)}(V), \phi^r)$ is isomorphic to each other. Thus we can choose a finite extension E' of E such that $(f_{E'}, r) = 1$, and the equality of Newton slopes of each factor follows from Lemma 3.2. \square

From the lemma it is also clear that V is crystalline iff $\dim_{E_0} D_{crys,r}^{(m)}(V) = \dim_{\mathbb{Q}_{p^r}} V$ for either m holds. There is a natural way to construct a ϕ -module from a ϕ^r -module. One starts with a ϕ^r -module (Δ, ψ) over E_0 and considers

$$D(\Delta, \psi) = \mathbb{Q}_p[t] \otimes_{\mathbb{Q}_p[t^r]} \Delta,$$

where Δ as $\mathbb{Q}_p[t^r]$ -module is given by $t^r(x) := \psi(x)$.

Lemma 3.11. *$D(\Delta, \psi)$ is naturally a ϕ -module over E_0 . Moreover, $D(D_{crys,r}^{(m)}(V), \phi^r)$ is isomorphic to $(D_{crys}(V), \phi)$.*

Proof. We define a σ -linear map ϕ on $D(\Delta, \psi)$ such that $\phi^r|_{\Delta} = \psi$. For $y = f(t) \otimes \lambda x \in D(\Delta, \psi)$, one defines $\phi(y) = tf(t) \otimes \lambda^\sigma x$. The formula

$$\phi^r(1 \otimes x) = t^r \otimes x = 1 \otimes t^r(x) = 1 \otimes \psi(x)$$

shows the requirement. To show the isomorphism in the statement, it suffices to show for $m = 0$. As E_0 -vector space, $D(D_{crys,r}^{(0)}(V), \phi^r)$ has a natural decomposition

$$\mathbb{Q}_p[t] \otimes_{\mathbb{Q}_p[t^r]} D_{crys,r}^{(0)}(V) = D_{crys,r}^{(0)}(V) \oplus \cdots \oplus t^{r-1} D_{crys,r}^{(0)}(V),$$

so that if $e = \{e_1, \dots, e_n\}$ is an E_0 -basis of $D_{crys,r}^{(0)}(V)$, then $t^m e$ is an E_0 -basis of $t^m D_{crys,r}^{(0)}(V)$ for $0 \leq m \leq r-1$. Note that $\phi^m e$ is an E_0 -basis of $D_{crys,r}^{(m)}(V)$. We defines an isomorphism of E_0 -vector spaces

$$f : \mathbb{Q}_p[t] \otimes_{\mathbb{Q}_p[t^r]} D_{crys,r}^{(0)}(V) \rightarrow D_{crys}(V)$$

by sending $t^m e_i$ to $\phi^m e_i$ for $1 \leq i \leq n$. Then one verifies that f is indeed an isomorphism of ϕ -modules by using the definition of the map ϕ on $D(D_{crys,r}^{(0)}(V), \phi^r)$ and the fact that $\phi^r|_{D_{crys,r}^{(0)}(V)}$ is the original ϕ^r -module structure on $D_{crys,r}^{(0)}(V)$. \square

This lemma means that, as far as the ϕ -module structure on $D_{crys}(V)$ is concerned, one suffices to study the ϕ^r -submodule $D_{crys,r}^{(m)}(V)$. Now we consider the filtration on $D_{dR}(V)$. Again from Lemma 3.9, we know that $(D_{dR}(V), Fil)$ is the direct sum of filtered submodules. Let

$$D_{dR,r}^{(m)}(V) := (B_{dR} \otimes_{\sigma^m, \mathbb{Q}_{p^r}} V)^{\text{Gal}_E},$$

and the induced filtration $Fil_m^i := Fil^i \cap D_{dR,r}^{(m)}(V)$ on $D_{dR,r}^{(m)}(V)$. Then one has

$$(D_{dR}(V), Fil) = \bigoplus_{m=0}^{r-1} (D_{dR,r}^{(m)}(V), Fil_m).$$

However we must warn that the Hodge slopes of individual factors (namely the Hodge slopes defined by Fil_m) can be *different*.

Proposition 3.12. *Let V be a crystalline \mathbb{Q}_{p^r} -representation. For each $0 \leq m \leq r-1$, one has*

$$(D_{crys,r}^{(m)}(V), \phi^r, Fil_m) = (D_{crys,r}^{(0)}(V_{\sigma^m}), \phi^r, Fil_0)$$

as filtered ϕ^r -modules.

Proof. By Lemma 3.7, $D_{crys}(V)$ is isomorphic to $D_{crys}(V_{\sigma^m})$ as filtered ϕ -modules. Now since

$$D_{crys,r}^{(0)}(V_{\sigma^m}) = (B_{crys} \otimes_{\mathbb{Q}_{p^r}} (\mathbb{Q}_{p^r} \otimes_{\sigma^m, \mathbb{Q}_{p^r}} V))^{\text{Gal}_E} = (B_{crys} \otimes_{\sigma^m, \mathbb{Q}_{p^r}} V)^{\text{Gal}_E} = D_{crys,r}^{(m)}(V),$$

the equality of the statement follows directly from the previous discussions. \square

Lemma 3.13. *Let V_1 and V_2 be two \mathbb{Q}_{p^r} -representations. One has a natural decomposition:*

$$V_1 \otimes_{\mathbb{Q}_p} V_2 \simeq \bigoplus_{m=0}^{r-1} V_1 \otimes_{\sigma^m, \mathbb{Q}_{p^r}} V_2.$$

In particular, $V_1 \otimes_{\mathbb{Q}_{p^r}} V_2$ is a direct factor of $V_1 \otimes_{\mathbb{Q}_p} V_2$.

Proof. It follows from Lemma 3.9. \square

Theorem 3.14. *Let V_1 be a crystalline \mathbb{Q}_p -representation and V a \mathbb{Q}_p -representation such that*

$$V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \simeq V_1 \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^{r-1}}.$$

Then one has the equality of filtered ϕ^r -modules:

$$(D_{crys}(V), Fil, \phi^r) = \otimes_{m=0}^{r-1} (D_{crys,r}^{(m)}(V_1), Fil_m, \phi^r).$$

If $\mathbb{Q}_{p^r} \subset E$, then the above equality is even an equality of filtered ϕ -module.

Proof. Put $\tilde{V} = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}$. One has

$$D_{crys,r}^{(0)}(\tilde{V}) = (B_{crys} \otimes_{\mathbb{Q}_{p^r}} (\mathbb{Q}_{p^r} \otimes_{\mathbb{Q}_p} V))^{\text{Gal}_E} = (B_{crys} \otimes_{\mathbb{Q}_p} V)^{\text{Gal}_E} = D_{crys}(V).$$

On the other hand, the tensor decomposition of \tilde{V} implies that

$$D_{crys,r}^{(0)}(\tilde{V}) = \otimes_{m=0}^{r-1} D_{crys,r}^{(0)}(V_{1,\sigma^m}),$$

and $D_{crys,r}^{(0)}(V_{1,\sigma^m})$ is equal to $D_{crys,r}^{(m)}(V_1)$ by Proposition 3.12 as filtered ϕ^r -modules.

Because of Lemma 3.10, $\otimes_{m=0}^{r-1} D_{crys,r}^{(m)}(V_1)$ is naturally a ϕ -module (with the map ϕ induced from the ϕ -structure on $D_{crys}(V_1)$). Now if $\mathbb{Q}_{p^r} \subset E$, then $\tilde{V} = V^{\oplus r}$ as Gal_E -modules. By Lemma 3.13, one sees that V is a direct factor of

$$V_1 \otimes_{\mathbb{Q}_p} V_{1,\sigma} \otimes_{\mathbb{Q}_p} \cdots \otimes_{\mathbb{Q}_p} V_{1,\sigma^{r-1}} = V_1^{\otimes r}.$$

Therefore, $D_{crys}(V)$ is a direct factor of $D_{crys}(V_1^{\otimes r}) = D_{crys}(V_1)^{\otimes r}$ as filtered ϕ -module. In particular, the ϕ -module structure on $D_{crys}(V)$ coincides with the restricted one from $D_{crys}(V)^{\otimes r}$. Write $D_{crys}(V_1) = \oplus_{m=0}^{r-1} D_{crys,r}^{(m)}(V_1)$. One sees immediately that $(\otimes_{m=0}^{r-1} D_{crys,r}^{(m)}(V_1), \phi)$ is a direct (filtered) ϕ -submodule of $D_{crys}(V_1)^{\otimes r}$. This concludes the proof. \square

Now we consider the case that V is polarisable and of Hodge-Tate weights 0, 1. Here V is polarisable means that there is a perfect Gal_E -pairing $V \otimes V \rightarrow \mathbb{Q}_p(-1)$. This condition implies that if λ is a Newton (resp. Hodge) slopes of V , then $1 - \lambda$ is also a Newton (resp. Hodge) slopes of V with the *same* multiplicity.

Theorem 3.15. *Let V be a polarisable crystalline representation with Hodge-Tate weights 0, 1. If there exists a two dimensional crystalline \mathbb{Q}_{p^r} -representation V_1 such that*

$$V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \simeq V_1 \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^{r-1}}$$

holds, then it holds that

- (i) *the Hodge slopes of V_1 is $\{2r - 1 \times 0, 1 \times 1\}$,*
- (ii) *the Newton slopes of V_1 is either $\{2r \times \frac{1}{2r}\}$ or $\{r \times 0, r \times \frac{1}{r}\}$.*

Consequently, there are only two possible Newton slopes for V : $\{2^r \times \frac{1}{2}\}$ or $\{1 \times 0, \dots, \binom{r}{i} \times \frac{i}{r}, \dots, 1 \times 1\}$.

Proof. Since the Hodge slopes of V is $\{n \times 0, n \times 1\}$, by Theorem 3.14, there exists a unique factor $D_{crys,r}^{(i)}(V_1)$ with two distinct Hodge slopes $\{0, 1\}$ and the other factors have all Hodge slopes zero. WLOG one can assume that $D_{crys,r}^{(0)}(V_1)$ has Hodge slopes $\{1 \times 0, 1 \times 1\}$ (and any other factor $\{2 \times 0\}$). Summing up

the Hodge slopes of all factors, one obtains the Hodge polygon of $D_{crys}(V_1)$ as claimed. By the admissibility of filtered ϕ -module on $D_{crys}(V_1)$, one finds that the Newton slopes of it must be of form $\{m_1 \times 0, m_2 \times \lambda\}$ where $m_1 + m_2 = 2r$ holds, and $\lambda \in \mathbb{Q}$ satisfies $\lambda m_2 = 1$. By Lemma 3.10, one finds that $r|m_i$, $i = 1, 2$. So $\frac{m_1}{r} + \frac{m_2}{r} = 2$ and $m_2 \neq 0$. There are two cases to consider:

Case 1: $m_1 = 0$. It implies that $m_2 = 2r$ and $\lambda = \frac{1}{2r}$.

Case 2: $m_1 \neq 0$. It implies that $m_1 = m_2 = r$ and $\lambda = \frac{1}{r}$. \square

4. NEWTON POLYGONS OF ABELIAN VARIETIES OF MUMFORD'S TYPE IN CHARACTERISTIC p

In this section we apply the previous results to study the Newton polygons of abelian varieties of Mumford's type. Let A be an abelian variety of Mumford's type defined over a number field K , and p a rational prime satisfying Assumption 2.2.

Lemma 4.1 (Noot, Proposition 2.1 [7]). *A has potentially good reduction at all places of K over p .*

Proof. The proof is that of Noot in the case of Mumford's example. Let us give a sketch only: Let \mathfrak{p} be a place of K over p . Choose a rational prime $l \neq p$ over which F is inert. By Deligne's absolute Hodge Proposition 2.9 [2], one has the factorization of the Galois representation

$$\rho_l : \text{Gal}_{K'} \rightarrow G(\mathbb{Q}_l) \rightarrow \text{GL}(H_{et}^1(\bar{A}, \mathbb{Q}_l))$$

for a finite extension K' of K . This implies the tensor decomposition

$$\rho_l \otimes \bar{\mathbb{Q}}_l = (\otimes_{i=1}^d \rho_i)^{\oplus 2^{\epsilon(D)}}.$$

Let v' be a place of K' over v and $I_{K',v'}$ the inertia group. The monodromy theorem of Grothendieck implies that $(\rho_l(\gamma) - id)^2 = 0$ for each $\gamma \in I_{K',v'}$. The factorization of $\rho_l \otimes \bar{\mathbb{Q}}_l$ implies that over $\bar{\mathbb{Q}}_l$ the matrix $\rho_l(\gamma)$ is conjugate to $(\otimes_{i=1}^d \rho_i(\gamma))^{\oplus 2^{\epsilon(D)}}$. Now that l is inert in F , $G^{ss}(\mathbb{Q}_l)$ is simple, and hence $\{\rho_i(\gamma)\}_{1 \leq i \leq d}$ have the same indices. Then $\rho_l(\gamma) = id$, $\forall \gamma \in I_{K',v'}$, and by Serre-Tate it has potentially good reduction. \square

Let \mathfrak{p}_K be a prime of K over p . By the above lemma we assume for simplicity that A has the good reduction A_k at \mathfrak{p}_K . Let $K_{\mathfrak{p}_K}$ be the completion of K at \mathfrak{p}_K , and $\rho : \text{Gal}_{K_{\mathfrak{p}_K}} \rightarrow \text{GL}(H_{et}^1(\bar{A}, \mathbb{Q}_p))$ the p -adic representation of the local Galois group. For simplicity we write $E = K_{\mathfrak{p}_K}$ and $H_{\mathbb{Q}_p} = H_{et}^1(\bar{A}, \mathbb{Q}_p)$. It is standard that $H_{\mathbb{Q}_p}$ is known to be a polarisable crystalline representation of Hodge-Tate weights $\{0, 1\}$.

Proposition 4.2. *Over an open subgroup $\text{Gal}_{E'}$ of Gal_E the representation ρ admits a natural tensor decomposition:*

$$H_{\mathbb{Q}_p} \simeq (V_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p})^{\oplus 2^{\epsilon(D)}},$$

where $U_{\mathbb{Q}_p}$ is an unramified representation. Consequently, both $V_{\mathbb{Q}_p}$ and $U_{\mathbb{Q}_p}$ are crystalline.

Proof. By Proposition 2.3 one has a tensor decomposition of ρ over an open subgroup $\text{Gal}_{E'}$:

$$H_{\mathbb{Q}_p} \simeq (V_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p})^{\oplus 2^{\epsilon(D)}}.$$

Recall that over \mathbb{Q}_p , one has

$$\text{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i=1}^n \text{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i).$$

It induces the presentation of $G(\mathbb{Q}_p)$ as a product:

$$G(\mathbb{Q}_p) = G_1 \times G_2,$$

where G_1 corresponds to the $\text{Gal}_{\mathbb{Q}_p}$ -orbit containing τ and G_2 the rest $\text{Gal}_{\mathbb{Q}_p}$ -orbits (hence G_2 may be trivial). By construction of V and U , it is clear that the projection of $\rho(\text{Gal}_{E'})$ to G_1 (resp. G_2)-factor acts on $U_{\mathbb{Q}_p}$ (resp. $V_{\mathbb{Q}_p}$) trivially. Now consider the Hodge-Tate cocharacter defined by the Hodge-Tate decomposition of $H_{\mathbb{Q}_p}$:

$$\mu_{HT} : \mathbb{G}_m(\mathbb{C}_p) \rightarrow G(\mathbb{C}_p) = G_1(\mathbb{C}_p) \times G_2(\mathbb{C}_p),$$

an the Hodge-de Rham cocharacter

$$\mu_{HdR} : \mathbb{G}_m(\mathbb{C}) \rightarrow G(\mathbb{C})$$

associated to the Hodge decomposition of $H_B^1(A(\mathbb{C}), \mathbb{Q})$. Let C_{HdR} (resp. C_{HT}) be the $G(\mathbb{C})$ (resp. $G(\mathbb{C}_p)$)-conjugacy class of μ_{HdR} (resp. μ_{HT}). Then C_{HdR} is defined over the reflex field of (G, X) which is $\tau(F) \subset \mathbb{C}$. The point is that there is a comparison (see Page 94 [8] and references therein):

$$C_{HT} = C_{HdR} \otimes_{F, \tau} \mathbb{C}_p.$$

Here one uses

$$\tau : F \rightarrow \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p \subset \hat{\bar{\mathbb{Q}}}_p = \mathbb{C}_p.$$

Hence one deduces that the projection of μ_{HT} to the second factor G_2 must be trivial since it is away from the τ -factor. By S. Sen's theorem (the \mathbb{Q}_p -Zariski closure of μ_{HT} is equal to $\rho(\text{Gal}_{E'^{ur}})$), the projection to G_2 of the inertia group $\subset \text{Gal}_{E'}$ under ρ is trivial. Thus $U_{\mathbb{Q}_p}$ is an unramified representation and it is then crystalline (cf. Proposition 7.12 [3]). As $V_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p}$ is a direct factor of $H_{\mathbb{Q}_p}$, it is crystalline, and therefore $V_{\mathbb{Q}_p}$, that is a subobject of $V_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p}^*$, is also crystalline. \square

The following result is known among experts. A variant of it was communicated by L. Berger to the first named author during the p -adic Hodge theory workshop in ICTP, 2009. The first official proof should appear in the Ph.D thesis of G. Di Matteo (see also recent preprint [4]). Another proof has been communicated to us by L. Xiao (see [10]).

Theorem 4.3. *Let V and W be two \mathbb{Q}_{p^r} -representations of Gal_E . If $V \otimes_{\mathbb{Q}_{p^r}} W$ is de Rham, and one of the tensor factors is Hodge-Tate, then each tensor factor is de Rham.*

Applying Theorem 4.3 to the tensor factor $V_{\mathbb{Q}_p}$ in Proposition 4.2, one obtains the following

Proposition 4.4. *Making an additional finite field extension $E' \subset E''$ if necessary, one has a further decomposition of $\text{Gal}_{E''}$ -representation:*

$$V_{\mathbb{Q}_p} \otimes \mathbb{Q}_{p^r} \simeq V_1 \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_r,$$

where $\text{Gal}_{E''}$ acts on \mathbb{Q}_{p^r} trivially and V_i is the σ^{i-1} -conjugate of V_1 . Then each tensor factor V_i is potentially crystalline.

Proof. Assume $r = 2$ for simplicity. The above tensor decomposition implies the tensor decomposition \mathbb{C}_p -representations:

$$V_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq (V_1 \otimes_{\mathbb{Q}_{p^2}} \mathbb{C}_p) \otimes_{\mathbb{C}_p} (V_{1,\sigma} \otimes_{\mathbb{Q}_{p^2}} \mathbb{C}_p).$$

Since $V_{\mathbb{Q}_p}$ is crystalline, it is Hodge-Tate. It implies that the Sen's operator Θ_V of $V_{\mathbb{Q}_p}$ is diagonalizable (over \mathbb{C}_p). Let Θ_{V_1} be the Sen's operator of V_1 . It can be written naturally as $\Theta_1 \oplus \Theta_{1,\sigma}$ where Θ_1 is associated to $V_1 \otimes_{\mathbb{Q}_p^2} \mathbb{C}_p$ and $\Theta_{1,\sigma}$ to $V_{1,\sigma} \otimes_{\mathbb{Q}_p^2} \mathbb{C}_p$. Thus one has

$$\Theta_V = \Theta_1 \otimes id + id \otimes \Theta_{1,\sigma}.$$

It implies that Θ_1 and $\Theta_{1,\sigma}$ are diagonalizable. Now consider the eigenvalues of them. For that we use the relation between the Hodge-Tate cocharacter and the eigenvalues of the Sen's operator: they are related by the maps \log and \exp . Continue the argument about Hodge-Tate cocharacter in Proposition 4.2. So let $\{\tau = \tau_1, \tau_2\}$ be the $\text{Gal}_{\mathbb{Q}_p}$ -orbit of τ . We can assume that in the above decomposition the V_1 -factor corresponds to τ . It follows that the projection of μ_{HT} to the $V_{1,\sigma}$ -factor is trivial. This implies that the eigenvalues of $\Theta_{1,\sigma}$ are zero. Particularly they are integral. So are those of Θ_1 . Hence Θ_{V_1} is diagonalizable with integral eigenvalues. So V_1 is Hodge-Tate, and by Theorem 4.3 it is de Rham. By the p -adic monodromy theorem, conjectured by Fontaine and firstly proved by Berger (see [1]), it is potentially log crystalline. One shows further that it is potentially crystalline. Let N_V (resp. N_{V_1}) be the monodromy operator of V (resp. V_1). Then as before, one has the formulas:

$$N_{V_1} = N_1 + N_{1,\sigma}, \quad N_V = N_1 \otimes id + id \otimes N_{1,\sigma}.$$

Since V is crystalline, $N_V = 0$. It implies that $N_1 = N_{1,\sigma} = 0$. Hence $N_{V_1} = 0$ and V_1 is potentially crystalline. \square

Now we can prove the main result of the paper.

Theorem 4.5. *Notation as above. Put $\mathfrak{p} = \mathfrak{p}_K \cap \mathcal{O}_F$ and $r = [F_{\mathfrak{p}} : \mathbb{Q}_p]$. Then the Newton polygon of A_k is either $\{2^{d+\epsilon(D)} \times \frac{1}{2}\}$ (i.e. supersingular) or*

$$\{2^{d-r+\epsilon(D)} \times 0, \dots, 2^{d-r+\epsilon(D)} \cdot \binom{r}{i} \times \frac{i}{r}, \dots, 2^{d-r+\epsilon(D)} \times 1\}.$$

Proof. The two decomposition results Proposition 4.2 and 4.4 show that the condition of Theorem 3.15 is satisfied for a finite field extension of E . Note also that the unramified representation $U_{\mathbb{Q}_p}$ contributes only the multiplicity 2^{d-r} to the Newton polygon. The theorem follows easily from Theorem 3.15. \square

The concluding paragraph discusses the existence of the possible Newton polygons. For the example of Mumford, the existence result was established by Noot (cf. §3-5 [8]) by studying the reductions of CM points in a Mumford's family. As a byproduct he has also obtained a classification of CM points in this case. There are divided into two cases: Let $F \subset L$ be a maximal subfield of D . Then L can either be written as $E \otimes_{\mathbb{Q}} F$ (E is necessary an imaginary quadratic extension of \mathbb{Q}) or not. To our purpose one finds the latter case generalizes, and the resulting generalization gives the necessary existence result. More precisely, Proposition 5.2 in loc. cit. provides the maximal subfields in D of the second case with the following freedom: Let \mathfrak{p} be a prime of F over p . Then L can be so chosen that \mathfrak{p} is split or inert in L . Secondly, Lemma 3.5 and Proposition 3.7 in loc. cit. work verbatim for a general D : Only one point shall be taken care of when the rational number b associated to D is not a square. In this case, one adds the multiplicity two to the constructions appeared therein. This step gives us an isogeny class of CM abelian varieties of Mumford's type. Finally the proof of Proposition 4.4, or rather the method of computing the Newton polygon for a CM abelian variety modulo a prime works in general. Thus one can safely conclude the existence result in the general case.

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